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# String Representation of Wilson Loops

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## Abstract

We explore the consequences of imposing Polyakov's zig/zag-invariance in the search for a confining string. We first find that the requirement of zig/zag-invariance seems to be incompatible with spacetime supersymmetry. We then try to find zig/zag-invariant string backgrounds on which to implement the minimal-area prescription for the calculation of Wilson loops considering different possibilities.

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# 1 Introduction

After the recent work of Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9] we have for the first time a way to look for confining string candidates by solving the loop equations of four-dimensional non-Abelian Yang-Mills theories. An important ingredient that such strings should satisfy is the so called zig/zag invariance: invariance under generic reparametrizations included those not preserving orientation of the parametrized loop [1]. Only after fulfilling this symmetry requirement we can achieve the main goal of satisfying the loop equations (see e.g. [10] and references therein). In this paper we will discuss some of the implications of zig/zag-invariance and its potential implementation in the framework of the near-horizon geometry of D-brane solutions.

A longstanding problem in non-Abelian gauge theories is to find a string representation of gauge non-local loop variables (Wilson loops), satisfying the Polyakov-Makeenko-Migdal loop equations [10]. It is known that fundamental strings, for instance Nambu-Goto strings, are not suited for this purpose owing to the fact, recently stressed by Polyakov [1], of their failure to reproduce the reparametrization invariance of the Wilson loop with respect to generic reparametrizations, including those with vanishing derivative at some point.

A Wilson loop is defined by

$$W(C) = \text{Tr } U(C) = \text{Tr } \left\{ P \exp i \oint_C A_\mu dx^\mu \right\}, \quad (1.1)$$

where the exponent is the integral of a one-form over a closed path and, as such, it enjoys invariance under *all* diffeomorphisms of the path, including those (orientation-reversing) corresponding to reparametrizations  $\xi \rightarrow \xi'$  where the Jacobian changes sign. This obviously implies:

$$W(C^{-1}) = W^{-1}(C). \quad (1.2)$$

Following Polyakov we will refer to this type of reparametrizations as zig/zag symmetry. The net geometrical effect of this type of reparametrizations is to induce foldings, hence a necessary condition for a string theory to be a reasonable candidate of Yang-Mills string or more generically of confining string, is to be blind to foldings [11].

This is a very strong requirement on the open string sector of the theory. In particular it means that all open string vertex operators, which include factors of the square root of the determinant of the worldsheet metric  $h_{\alpha\beta}$  should decouple. This is equivalent to say that small variations of the loop should be saturated, in the corresponding string representation, by open string vertex operators creating states in the first Virasoro level, which are the ones representing the massless gluons.

Those at higher levels, which represent massive states of higher spin, are associated with vertex operators that must be integrated over the boundary of the worldsheet using  $\sqrt{h}$ .

If some of this higher-level modes contribute to the variation of the Wilson loop, then we necessarily lose zig/zag invariance (due to the dependence of the vertex operator on the

worldsheet metric) and very likely we will find terms in the loop variation not consistent with the Yang-Mills loop equations.

The simplest and more radical way to guarantee zig/zag invariance is, of course, to require the string-induced metric on the boundary of any open string worldsheet amplitude to vanish. In fact in this case any vertex operator that potentially can spoil zig/zag invariance is automatically decoupled from all possible open string amplitudes. A question immediately arises: does the truncation imposed by requiring zig/zag invariance, produce a consistent string theory?

We can now imagine a string metric background  $G$  with “horizons” to be defined as subspaces of spacetime with vanishing pullback metric. We could require these “horizon” subspaces to contain time. In this case the condition of zig/zag invariance for a string living in this particular spacetime is tantamount to impose Dirichlet boundary conditions on the coordinates transverse to the “horizon”. We refer to these “horizons” as *zig/zag horizons*.

Of course, the string metric  $G$  is not arbitrary and must satisfy the generic constraints imposed by conformal invariance. So the first problem we face in order to find a confining string is to discover a consistent string metric, possessing a four dimensional spacetime horizon to be identified with the spacetime where our Yang-Mills theory is going to be defined.

Following Polyakov we will identify the transversal coordinates with the Liouville field. The requirement of zig/zag invariance amounts to impose Dirichlet boundary conditions on the said Liouville coordinate.

It is important here to stress the difference between what we just called zig/zag horizons and Dirichlet branes and near-horizon geometries.

In a zig/zag horizon we impose Dirichlet boundary conditions on the horizon in very much the same way as we do it for standard D-branes. The main difference with D-branes is that now we impose the vanishing of the string metric in the horizon in order to project out the open string higher level modes, something that clearly we do not impose in standard D-brane physics.

On the other hand, in the near-horizon approach to D-branes using Maldacena’s limit, massive modes are decoupled but the spacetime picture of a D-brane is lost and replaced by Supergravity on a certain  $AdS_5 \times S_5$  space-time.

It should be kept in mind that zig/zag horizons, being the locus for Dirichlet boundary conditions, become in the same way as for generic D-branes, a source for gravitons. In fact it would be in principle possible to compute, following Polchinski’s prescription, some sort of zig/zag-brane tension. Most likely zig/zag-branes could be simply thought of as candidates for non-critical D-branes.

## 2 Zig/Zag Invariance and Supersymmetry

In our previous discussion we simply addressed the question of zig/zag invariance for the simplest bosonic case. A question that arises immediately is the consistency of zig/zag

invariance and supersymmetry.

In the discussion of this issue it is important to separate worldsheet supersymmetry and spacetime supersymmetry. The worldsheet supersymmetric Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \sqrt{h} \left\{ h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{2} \psi^\mu \gamma^\alpha \nabla_\alpha \psi_\mu \right. \\ & \left. + \frac{i}{2} (\chi_\alpha \gamma^\beta \gamma^\alpha \psi^\mu) (\partial_\beta \chi_\mu - \frac{i}{4} \chi_\beta \psi_\mu) \right\} , \end{aligned} \quad (2.3)$$

which implies that the zig/zag condition of tensionless open strings does not clash with worldsheet supersymmetry. More precisely the  $\sqrt{h}$  factor, that spoils zig/zag invariance is common to both the boson and fermionic worldsheet sectors. The only bosonic vertices allowed by zig/zag invariance are the ones for massless vector bosons

$$\oint DX e^{ik \cdot X} d\xi d\theta , \quad (2.4)$$

with  $X$  being the string coordinate superfield and  $D$  the worldsheet superderivative. In order to discuss spacetime supersymmetry we should consider fermionic emission vertices. As it is well known the definition of these vertices in the NSR formalism requires the use of picture-changing manipulations [12]. In particular, for the open string the fermion vertex of conformal dimension 1 is, in the  $-1/2$  picture, given by

$$\oint \sqrt{h} e^{-\frac{\phi}{2}} S^\alpha e^{ik \cdot X} d\xi , \quad (2.5)$$

where  $\phi$  is the field introduced in the bosonization of the ghost current and  $S$  is the  $SO(10)$  spinor. From the preceding expression we observe that zig/zag invariance, at least naively, is not consistent with spacetime supersymmetry, i.e. with the existence of non-vanishing open-string amplitudes for fermion vertex operators. A way to see the potential difficulties for compatibility between zig/zag invariance and spacetime supersymmetry is already suggested by the asymmetric dependence on the worldsheet metric of the two terms involved in the  $\kappa$ -invariant worldsheet action, where the extra Wess-Zumino term is manifestly independent on the worldsheet metric and, therefore, zig/zag invariant.<sup>2</sup>

A different issue we can worry about is the fate of tachyons in zig/zag-invariant strings. It is interesting to observe that both in the standard bosonic case and in the NSR superstring, before imposing GSO projection, the tachyon vertex operators are not consistent with zig/zag invariance, since in both cases they depend on the determinant of the worldsheet metric  $h_{\alpha\beta}$

$$\left\{ \begin{array}{l} \oint \sqrt{h} e^{ik \cdot X} \Big|_{k^2=-2} , \\ \oint \sqrt{h} k \cdot \psi e^{ik \cdot X} \Big|_{k^2=-1} , \end{array} \right. \quad (2.6)$$

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<sup>2</sup>In the light-cone gauge the fermionic vertex operator is like the one in equation (2.5) without the spinor ghost and with an  $SO(8)$  spinor field that behaves as a  $1/2$  spinor in two dimensions, which makes clear the dependence on the determinant of the metric in the integrated version

and therefore they should decouple by the zig/zag mechanism.

In summary, what we conclude from our previous discussion is that a confining string satisfying zig/zag invariance, even if it is world-sheet supersymmetric, is truncating the open string spectrum to pure gluons only.

As mentioned above, we can try to understand this phenomenon from the point of view of  $\kappa$ -symmetry. In fact, space-time supersymmetry amounts to consider loops in superspace and to work directly with the Green-Schwarz superstring. Imposing zig/zag invariance in the Green-Schwarz superstring is equivalent to reduce ourselves to the topological Wess-Zumino term in what concerns the open string sector, which in turn implies that we lose  $\kappa$ -invariance.

### 3 Confining Strings and Near Horizon Geometry

Polyakov's Ansatz for confining strings is based on a non-critical string in four dimensions, supplemented by a Liouville field  $\varphi$  so the effective 5-dimensional metric has the form [1]

$$ds^2 = a^2(\varphi)dx_{\parallel}^2 - d\varphi^2, \quad (3.7)$$

satisfying the horizon condition:

$$a(\varphi_0) = 0. \quad (3.8)$$

Zig/zag symmetry is implemented by requiring the gauge fields to live on the horizon. This is equivalent to impose the following Ansatz for the Wilson loop:

$$W(C) = \int \mathcal{D}X \mathcal{D}\varphi \exp \left\{ - \int_{\Sigma} G_{\mu\nu}(X, \varphi) \partial X^{\mu} \bar{\partial} X^{\nu} + T(X, \varphi) \partial \varphi \bar{\partial} \varphi + \dots \right\}. \quad (3.9)$$

where one integrates over all embeddings  $X^{\mu}$  of the worldsheet whose boundary is on the horizon  $\varphi = \varphi_0$  and is the loop  $C$ . In this expression the part of the metric Eq. (3.7) parallel to the brane (the 4-dimensional spacetime where we want to calculate Wilson loops),  $a^2 dx_{\parallel}^2$  has been rewritten as  $G_{\mu\nu} dX^{\mu} dX^{\nu}$ . The consistency conditions on the metric will be the ones implied by conformal invariance of the closed string sector.

A very different computation of the Wilson loop based on confining strings, inspired on the holographic map and on previous work on absorption coefficients for non-dilatonic D-branes [2] was developed by Rey, Yee and Maldacena [3, 4] and further extended by Witten [6]. In that approach the spacetime metric that enters the string sigma model is the one that describes the near-horizon geometry of the D-3-brane:  $AdS_5 \times S^5$ , which is assumed to lead to a conformally-invariant theory. The  $AdS_5$  part

$$ds^2 = \frac{R^2}{z^2} (dx_{\parallel}^2 - dz^2) - R^2 d\Omega_5^2, \quad (3.10)$$

plays the most important role. It can be written in the form of Eq. (3.7) with the change of variables

$$z = Re^{\varphi/R}. \quad (3.11)$$

but the fifth coordinate does not have a Liouville field interpretation. Furthermore, they proposed to place the loop  $C$  in the *far from the brane region* of  $AdS$  spacetime (that is: far from the zig/zag horizon). A semiclassical approximation to the Wilson loop is given by the minimal area surface in  $AdS_5$  whose boundary is the loop  $C$ . Renormalization is necessary and it corresponds to the subtraction of the length of the loop in the  $AdS_5$  metric at the far-from-the-brane boundary region.

This definition of the Wilson loop is manifestly not zig/zag-invariant since the pullback metric at the boundary space where the loop is located is not vanishing. Notice that the problem with zig/zag-invariance is not only due to the modified Wilson loop, used in [3, 4], that includes the quantum numbers associated with the transversal coordinates in  $S^5$ , but is a problem even for the finite-temperature models where supersymmetry is explicitly broken and effective decoupling of Higgs fields is assumed, as it is done in [6, 9].

We can now compute the semiclassical approximation to the zig/zag-invariant Wilson loop for the  $AdS_5$  confining string. For simplicity we will consider a rectangular Wilson loop in spacetime with  $\ell$  being the separation between the static *quarks* (actually, infinitely heavy gauge bosons). The zig/zag-invariant Wilson loop is then given on the horizon by

$$\begin{cases} \tau = X^0, \\ \sigma = X, \end{cases} \quad (3.12)$$

where  $X$  is one of the spacelike coordinates and  $\sigma \in [-\ell/2, +\ell/2]$ ,  $\tau \in [0, T]$ .

The semiclassical approximation will be determined by the minimal area surface corresponding to the profile  $\varphi = \varphi(\sigma)$  (with the boundary conditions  $\varphi(\pm\ell/2) = \varphi_0$ ). The induced metric on the worldsheet is given by

$$ds^2 = a^2 d\tau^2 - (a^2 - \varphi_\sigma^2) d\sigma^2. \quad (3.13)$$

where  $\varphi_\sigma \equiv d\varphi/d\sigma$ .

The corresponding Nambu-Goto action will then be the integral of the determinant of the induced metric, that is:

$$S = T \int_{-\ell/2}^{\ell/2} d\sigma \, a \sqrt{a^2 + \varphi_\sigma^2}. \quad (3.14)$$

The *energy* associated to translations in  $x$  is given by:

$$E = -\frac{a^3}{\sqrt{a^2 + \varphi_\sigma^2}}. \quad (3.15)$$

The most general solution to this problem is a string “ironed” over the horizon:

$$a = 0. \quad (3.16)$$

It is easy to prove that this result is general for any Polyakov confining string Ansatz, because from energy conservation we easily get:

$$\varphi_\sigma^2 = a^2 \left( \frac{a^4}{E^2} - 1 \right), \quad (3.17)$$

constraining the motion to lie on the region  $a^2 > E$ . The only solution which can be attached to the horizon at the endpoints  $\sigma = \pm \ell/2$  is the trivial one  $a = 0$ .

## 4 Exploring other Possibilities

In this section we will explore other ways to implement this set of ideas different from using  $AdS_5$  as a string background (supplemented by an  $S_5$  “hidden sector”).

The first thing one can try is to find directly solutions of the  $\beta$ -functions (with no hidden sector) equations satisfying all the requirements. Thus, we can look for solutions of the equations of motion corresponding to the action

$$S = \int d^d x \sqrt{|g|} e^{-2\phi} \left[ R - 4(\partial\phi)^2 + \frac{1}{2 \cdot 3!} H^2 + \frac{(d-26)}{3\alpha'} \right], \quad (4.18)$$

of the form (3.7). These solutions necessarily have a non-trivial dilaton field, due to the dilaton potential proportional to the central-charge deficit. We will not consider non-trivial Kalb-Ramond fields. The simplest solutions are, then, domain-wall-type solutions. These have been intensively studied in Refs. [14] for the generic model

$$S = \int d^d x \sqrt{|g_E|} \left[ R_E + \frac{1}{2} (\partial\varphi)^2 + \frac{1}{2 \cdot 3!} H^2 + \frac{1}{2} m^2 e^{-a\varphi} \right]. \quad (4.19)$$

Real solutions exist in any dimension for strictly negative values of the parameter

$$\Delta = a^2 - 2 \frac{(d-1)}{(d-2)}. \quad (4.20)$$

If we reverse the sign of the scalar potential (or, equivalently we make  $m$  imaginary), then, solutions exist for any strictly positive value of  $\Delta$ . If we rescale our action to the Einstein frame and then we canonically normalize the dilaton field, as above,, we find that our case is mixed, and we can only find real solutions if we use as coordinate transverse to the domain wall a *timelike* coordinate. If  $x$  is such a coordinate, then, the solution looks in the Einstein frame

$$\begin{cases} ds_E^2 &= H^2 [dt^2 - d\vec{y}_{d-2}^2] + dx^2, \\ e^\phi &= H^{-\frac{(d-2)}{2}}, \end{cases} \quad (4.21)$$

where  $H = bx + c$ , where the constant  $b$  is a function of the dimension,  $m$  and the parameter  $a$  and  $c$  can be set to zero by a coordinate shift. Rescaling the solution back to the string frame and after a simple coordinate change we find

$$\begin{cases} ds^2 &= dt^2 - d\vec{y}_{d-2}^2 + dz^2, \\ e^\phi &= e^{-\frac{(d-2)}{2}H}. \end{cases} \quad (4.22)$$

This solution is similar to the D-instanton solution in the sense that both have a flat spacetime. Precisely for this reason, it is totally inappropriate for our purposes.

With our Ansatz, the above solution is unique. Considering a non-trivial 2-form field would privilege a 2-dimensional subspace and this would not be consistent with the Ansatz. Thus, we have to look for other kinds of solutions, now with an appropriate hidden sector that makes zero the total central charge.

First, it is easy to see, that there are no vacuum solutions (apart from Minkowski) of the string  $\beta$ -functions of the form (3.7). Thus, it is natural to explore other possibilities relaxing some of the conditions that the string background has to satisfy.

If we relax the condition that the function  $a$  that vanishes on the horizon is a common factor for the metric of the whole 4-dimensional space, we find a vacuum solution which is unique and depends on only one integration constant:

$$ds^2 = a^2 dt^2 - d\vec{y}_3^2 \pm d\varphi^2, \quad a = 1 + b\varphi. \quad (4.23)$$

After a straightforward calculation we find that the action is, up to a numerical coefficient

$$S \sim T b \ell^2. \quad (4.24)$$

This background solves the  $\beta$ -functions and leads to the area law but it does not respect zig/zag-invariance. This result is independent of the dimension (unless the constant  $b$  contains information about it in a way unknown to us).

The next possibility is to add a non-trivial dilaton. Using Polyakov's Ansatz (3.7) we find in  $d$ -dimensions a unique solution

$$ds^2 = a^{2\alpha} (dt^2 - d\vec{y}_3^2) \pm d\varphi^2, \quad e^{\phi - \phi_0} = a^\beta, \quad (4.25)$$

where  $a$  has the same form as in the previous solution and  $\alpha, \beta$  are two numerical constants that depend on the dimension:

$$\begin{cases} \alpha &= 1 - \frac{(d-2)}{(d-1)} \frac{1}{1 \pm \sqrt{\frac{2}{(d-2)(d-1)}}}, \\ \beta &= \pm \sqrt{\frac{2(d-2)}{(d-1)}} \left[ \frac{1}{(d-1)} \pm \sqrt{\frac{2}{(d-2)(d-1)}} \right]. \end{cases} \quad (4.26)$$

(The two possible signs in these constants are unrelated to the two possible signatures).



As we explained in the Introduction, there is no solution for the string action in this background.

Again, we have to relax the conditions satisfied by our metric. There is a whole family of solutions that we can now use but we will focus in a particular one: the T dual of the vacuum solution Eq. (4.23) in the time direction:

$$ds^2 = a^{-2}dt^2 - d\vec{y}_3^2 \pm d\varphi^2, \quad e^{\phi-\phi_0} = a. \quad (4.27)$$

Now there is solution only for the lower sign:

$$a = b\sqrt{(\ell/2)^2 - \varphi^2}, \quad S \sim \frac{1}{b^3\ell^2}, \quad (4.28)$$

which doesn't give the are law.

We would like to stress two facts with respect to this solution:

1. The behavior is completely different from that of its T dual. This may be yet another example of T duality not working in the time direction.
2. The sign of the dilaton field can be reversed  $\phi'$ . This allows us to fit the Yang-Mills  $\beta$ -function:

$$\beta = \frac{\partial g}{\partial \log \mu} = \frac{\partial e^{\phi'/2}}{\partial \varphi} \sim a^{-3/2} \sim e^{3\phi'/2} \sim g^3. \quad (4.29)$$

## 5 Liouville Dressing

A different attempt to find a confining solution enjoying zig/zag-invariance would imply a modification, by introducing a Liouville potential, of the worldsheet Lagrangian on the near-horizon geometry for D-3-branes<sup>3</sup>.

The logic behind this modification is as follows. In the case of D-3-branes the geometry near the horizon is  $AdS_5 \times S^5$  with the  $S^5$  parametrizing the extra scalars required for  $N = 4$  supersymmetry.

On the other hand, in the original Polyakov approach to confining strings, a non-critical string with four spacetime coordinates and one extra Liouville coordinate is used. This Liouville coordinate is constrained by the condition of vanishing  $\beta$ -functions, which, in principle, can be implemented by using some string effective action, and by the requirement of vanishing central extension.

A potential zig/zag interpretation of the  $AdS_5 \times S^5$  near-horizon metric amounts to identify the radial coordinate with the Liouville field as done in Eq. (3.11).

However in order to make more precise this identification we should correct the Liouville field in a way that effectively mimics the contribution to the central extension of the extra

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<sup>3</sup>In the Liouville framework we are forced to consider “non-critical” D-branes with the Liouville field in the transverse direction. It would be interesting to work out, for these non-critical D-branes, the effect of tadpoles for macroscopic string states [15].

coordinates parametrizing the  $S^5$ . The most naive way to achieve that is to use a Liouville field with central extension  $c_L = 6$ . In the conformal gauge such a Liouville field is defined by

$$S_{\text{Liouville}} \equiv \frac{1}{4\pi\gamma} \int d\sigma d\tau \left\{ \frac{1}{2} [(\partial_\tau \varphi)^2 - (\partial_\sigma \varphi)^2] - \mu^2 e^{2\varphi} \right\}. \quad (5.30)$$

The classical central extension is given by  $c = 3/\gamma$ . If we rewrite  $c = 6$  we get  $\gamma = 1/2$ . If, instead, we use the quantum central extension  $c = 3 + 1/\gamma$  then we must set  $\gamma = 1/3$ . This difference will not be relevant for our qualitative discussion of the Wilson loop and we will use the classical value  $\gamma = 1/2$  only. The factor  $\mu^2$  in the above action is a worldsheet cosmological constant term. As we are going to see in a moment, non-trivial solutions for zig/zag-invariant Wilson loops require the cosmological constant to be negative (as above).

In a certain sense the addition of the Liouville potential amounts to include a sort of gravitational dressing to the Wilson loop, preserving zig/zag-invariance. The roughest approximation we can think of for the computation of Wilson loops for static quarks will then consist in taking the  $AdS$  metric and simply adding the Liouville potential

$$S = \frac{1}{4\pi} \int d\sigma d\tau [a^2(\phi)(\partial x_{\parallel})^2] + S_{\text{Liouville}}, \quad (5.31)$$

where

$$a(\varphi) = e^{-\varphi/R}. \quad (5.32)$$

Observe that we are identifying the  $\varphi$  spacetime coordinate with the Liouville field and the natural scale for the spacetime coordinate is the  $AdS$  radius  $R$  so that is also the natural scale for the Liouville. In what follows we will work in units in which  $R = 1$ .

Notice also that we are working in the conformal gauge where the Liouville mode is precisely the conformal factor of the worldsheet metric, so that instead of minimizing the Nambu-Goto action plug our Ansatz into the above action and look for solutions of the resulting equations of motion. There is also a first integral given by:

$$E = -\frac{1}{2}\gamma(\partial_\sigma \phi)^2 - 2a^2 + \frac{\mu^2}{\gamma}e^{2\phi} \quad (5.33)$$

The solution satisfying the zig/zag constraints is given in terms of the following elliptic integral

$$x - \ell/2 = -\frac{1}{\mu\sqrt{2}\sqrt{t_-^2 + t_+^2}} \mathbf{ds}^{-1} \left( \frac{e^\phi}{\sqrt{t_-^2 + t_+^2}} \middle| \frac{t_-^2}{t_-^2 + t_+^2} \right), \quad (5.34)$$

where  $\mathbf{ds}$  is the Jacobi elliptic function which has a zero at the point  $K + iK'$  (where  $K$  and  $K'$  are respectively the real and imaginary quarter periods) and a pole at the origin.  $t_+^2$  and  $-t_-^2$  are the solutions of the equation

$$t^4 - \frac{\gamma E}{\mu^2} t^2 - \frac{2\gamma}{\mu^2} = 0. \quad (5.35)$$

Giving the fact that  $4K$  is a real period of the Jacobi function, where  $K$  is given in terms of the *modulus*  $m \equiv \frac{t_-^2}{t_-^2 + t_+^2}$  by

$$K(m) \equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}} \quad (5.36)$$

we can enforce another zero at  $x = -\ell/2$  by equating

$$4 \frac{K(m)}{\mu \sqrt{2} \sqrt{t_-^2 + t_+^2}} = \ell \quad (5.37)$$

which gives the *energy*  $E$  as a function of  $\ell$ .

In the limit  $E \rightarrow \infty$   $t_+ \sim \gamma E$  and  $t_- \sim 0$ . This limit corresponds to  $\ell = \frac{2\pi}{\sqrt{2\gamma E}}$ , up to the point that all our approximation can not be trusted anymore.

Using the fact that the Jacobi function in this limit (for zero modulus) reduces to  $\frac{1}{\sin}$ , we get for the action the following result

$$W(C) = \frac{-2\pi^2 T}{\gamma \ell}, \quad (5.38)$$

If we were to cut off the integral over  $\varphi$  at  $\varphi = 0$  nothing would change much. This can be easily seen to be equivalent to a maximum allowed value for the energy  $E$ .

This is a non-confining solution that very likely reflects the fact that we are working with *AdS* spacetime. The effect of the Liouville mode simply reduces to allow a zig/zag invariant solution.

Physically, thus, the Liouville term with negative cosmological constant acts as a sort of “potential well” allowing the existence of “bound states” in a mechanism somewhat similar to the one proposed by Witten in his black-hole-type solutions [7].

It is worth noticing that the potential problems concerning quantum Liouville theory would be important for the closed string sector in the bulk. For the open string sector we can expect the effect of Liouville tachyons to be suppressed on the horizon as a consequence of zig/zag-invariance (the metric on the horizon goes to zero as the inverse of the Liouville potential).

## 6 Conclusion

The aim of this paper has been to investigate Polyakov’s approach to confining string representations of Wilson loops, based on imposing zig/zag invariance. Next we summarize the main conclusions of our research.

First of all we have observed, by working out several examples, that the conditions of zig/zag-invariance impose very severe constraints on the spacetime metrics. In fact,

without modifications, either involving a non-trivial dilaton background, or some type of Liouville dressing, we have been unable to find a satisfactory answer. This situation contrasts with the potentially easy task of finding good string representations for Wilson loops, even in the case in which supersymmetry is broken by the mechanism of Witten [7], once the condition of zig/zag invariance is relaxed.

Thus the whole problem boils down to understand the physical relevance of such strong reparametrization invariance requirement. It seems clear that only zig/zag-invariant Wilson loops have a chance to satisfy Makeenko-Migdal-Polyakov loop equations. If this condition is implemented by forcing the Yang-Mills fields to live on some horizon hypersurface, then we are forced to modify drastically the holographic map, if we want to make it consistent with generic zig/zag reparametrizations. Geometrically, in the context of near-horizon D-brane geometries, this turns out to be equivalent to define the four-dimensional theory on the horizon and not on the boundary as it is compulsory in the holographic approach.

Zig/zag-invariance is, however, an ingredient that strongly affects the definition of a consistent string theory. In particular, if we reduce ourselves to the bosonic string the condition of zig/zag changes the way we should interpret the tachyon instability. In fact if we succeed in obtaining a bosonic zig/zag invariant string the tachyon vertex would not appear since it is not zig/zag-invariant. In other words, this general-reparametrization invariance acts as a projector that automatically suppress the tachyon from the bosonic string spectrum.

This issue can be potentially interesting since the critical dimension  $d = 26$  of the bosonic string is already magically attached to the coefficient of the pure Yang-Mills  $\beta$ -function in  $N = 0$ . Namely the one-loop Yang-Mills  $\beta$ -function [16]

$$\beta = -\frac{g^3}{16\pi^2} \frac{C_2(G)}{6} [22 - 4\nu(M) - \nu(R)] , \quad (6.39)$$

(where  $\nu$  is the number of Majorana fermions ( $M$ ) or real scalar fields ( $R$ )) vanishes in the case in which there are 22 scalar and no fermion fields. which are precisely the number of transversal coordinates for a (bosonic) D-3-brane in 26 dimensions.

The reason nobody cares about this purely  $N = 0$  theory defined on the world volume of a bosonic D-3-brane is of course because of the tachyon instability. It is precisely at that point where the relevance of zig/zag symmetry as a potential way to project out the tachyon, for the open string sector, may become relevant.

The other question we have observed is that zig/zag-invariance seems to differentiate between the worldsheet supersymmetry and the spacetime supersymmetry. As discussed above this is due to the form of fermionic vertex operators. Assuming that for a moment it seems that the requirement of zig/zag invariance not only projects out the open string tachyons but also “breaks” spacetime supersymmetry. This “breaking” can be considered dynamical only if we have some dynamical procedure to implement the zig/zag invariance. In this paper we have suggested a wild and, needless to say, very primitive possibility, based on the observation that the metric on the world volume for the supergravity solutions for D-branes is zero, and in that way a candidate to zig/zag invariant metric.

Finally, a last comment on our Liouville approach and  $N = 0$  solutions. In order to go to  $N = 0$  the most natural approach is, following Witten's suggestion, to decouple the extra matter in the  $N = 4$  supermultiplets. The Liouville attempt consists in replacing the geometry used in performing this decoupling by simply introducing a Liouville mode contributing in the appropriated way to the central charge. We can even imagine that once we move in the Liouville direction (actually going from the horizon to the boundary) we reach a region where the Liouville mode is effectively decoupled, interpolating in a certain way between the zig/zag invariant horizon region with  $N = 0$  and a boundary region with conformal invariance.

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## A Global Structure of $AdS$ and $CAdS$

Given the basic importance of Anti de Sitter metric in the whole description of the space-time region close to the brane, we have collected here some geometric facts, relevant for the discussion of boundary conditions in the main text.

Anti de Sitter space in  $p$  dimensions ( $AdS_p$ ) is defined ([17]) as the induced metric on the hyperboloid  $(X^0)^2 + (X^p)^2 - \delta_{ij}X^iX^j = 1; (i, j = 1 \dots p-1)$  embedded in an ambient space  $\mathbb{R}_{2,p-1}$  (that is,  $\mathbb{R}_{p+1}$  endowed with a Minkowskian metric with two times,  $ds^2 = (dX^0)^2 + (dX^p)^2 - \delta_{ij}dX^i dX^j$ ). Defined in that way, it clearly has topology  $S^1 \times \mathbb{R}^{p-1}$  (as well as closed timelike curves). The universal covering space ( $CAdS_p$ ) has topology  $\mathbb{R}^p$ .

This definition makes it manifest the underlying  $O(2, p-1)$  symmetry. As it is well known, there is a  $2-1$  correspondence between  $O(2, p-1)$  and the conformal group of Minkowski space in  $p-1$  dimensions,  $C(1, p-2)$ .

The *boundary* at infinity  $\partial AdS$  can be defined as the region where all  $X^\mu$  are rescaled by an infinite amount,  $X^\mu \rightarrow \xi X^\mu$ , where  $\xi \rightarrow \infty$ . In that way, the boundary is characterized by the relationship  $(X^0)^2 + (X^p)^2 - \delta_{ij}X^iX^j = 0$ , which is nothing but the well-known  $O(2, p-1)$  null-cone compactification of Minkowski space,  $M^\mathbb{C}$ . [18]. The way it works is that to any regular point of Minkowski space,  $x^\mu \in M$ , there corresponds another point in  $M^\mathbb{C}$ , namely

$$\left\{ \begin{array}{l} X^0 = x^0, \\ X^i = x^i, \\ X^{p-1} = \frac{1+x^2}{2}, \\ X^p = \frac{1-x^2}{2}. \end{array} \right. \quad (\text{A.40})$$

The points in  $M^{\mathbb{C}}$  which are not in  $M$  correspond to  $X^p + X^{p-1} = 0$ . This means that this compactification amounts to add an extra null cone at infinity.

The  $AdS$  metric can be easily put in the *globally static form* by means of the ansatz

$$\left\{ \begin{array}{l} X^0 = \cos \tau \cosh \chi, \\ X^p = \sin \tau \cosh \chi, \\ X^i = \sinh \chi n^i, \end{array} \right. \quad (\text{A.41})$$

where  $\delta_{ij} n^i n^j = 1$ ,  $(i, j = 1, \dots, p-1)$ . The result is

$$ds^2 = (\cosh \chi)^2 d\tau^2 - (d\chi)^2 - (\sinh \chi)^2 d\Omega_{p-2}^2. \quad (\text{A.42})$$

$AdS$  corresponds to  $0 \leq \tau \leq 2\pi$ , and  $CAdS$  to  $0 \leq \tau \leq \infty$

The antipodal map  $J : X \rightarrow -X$ , corresponds in this coordinates simply to  $(\tau, \chi, \vec{n}) \rightarrow (\tau + \pi, \chi, -\vec{n})$ .

A different, but closely related set of coordinates are the ones used by Susskind and Witten [17]. The metric has the form

$$ds^2 = \frac{R^2}{(1-r^2)^2} (4 \sum_{i=1}^{p-1} (dx^i)^2 - (1+r^2)^2 dt^2). \quad (\text{A.43})$$

They are easily obtained from the globally static form by

$$\sinh \chi = \frac{2r}{1-r^2}. \quad (\text{A.44})$$

$CAdS$  itself corresponds to the ball  $r < 1$ , and the boundary sits on the sphere  $r = 1$ .

Another interesting set of coordinates (common to all constant curvature spaces) is Riemann's, in which the metric reads

$$ds^2 = \frac{\eta_{\mu\nu} dy^\mu dy^\nu}{(1 - \frac{r^2}{4R^2})^2}, \quad (\text{A.45})$$

where  $\mu, \nu, 0, \dots, p-1$ ,  $\eta_{\mu\nu}$  is the ordinary Minkowski metric, and  $r^2 \equiv \eta_{\mu\nu} y^\mu y^\nu$ .

This coordinates are quite natural in the following sense. They can be constructed ([19]) by

$$y^\mu \equiv 2Ru^\mu \tanh \frac{s}{2R}, \quad (\text{A.46})$$

where  $u^\mu$  is the unit tangent vector to the geodesic going to the point P from a fiducial point  $P_0$ ; and  $s$  is the geodesic distance from  $P_0$  to  $P$ , and using the fact that the geodesic deviation between neighboring geodesics grows as  $\eta = R(\frac{d\epsilon}{ds})_{s=0} |\sinh \frac{s}{R}|$ , and that the angle between the tangents to such neighboring geodesics is precisely the volume element on the unit Minkowskian sphere, which can be easily obtained in terms of the ordinary volume on the unit Euclidean sphere:  $d\Omega_{p-1}^2(\text{hyperbolic}) \equiv -d\xi^2 - \sinh^2 \xi d\Omega_{p-2}^2$ .

In that way

$$ds^2 = dr^2 - R^2 \sinh^2 \xi \frac{r}{R} (d\xi^2 + \sinh^2 \xi d\Omega_{p-2}^2). \quad (\text{A.47})$$

The *horospheric coordinates* used by Gubser, Klebanov and Polyakov [5] are defined as

$$\left\{ \begin{array}{lcl} X^0 & = & t/z, \\ X^a & = & x^a/z, \\ X^p - X^{p-1} & = & 1/z, \end{array} \right. \quad (\text{A.48})$$

(where  $a = 1, \dots, p-2$ )

The metric reads

$$ds^2 = \frac{1}{z^2} (dt^2 - d\vec{x}_{p-2}^2 - dz^2), \quad (\text{A.49})$$

where  $d\vec{x}_{p-2}^2$  is the Euclidean line element in  $\mathbb{R}_{p-2}$ .

The metric above enjoys a manifest  $O(1, p-2)$  symmetry. Besides, it is invariant under *dilatations*  $x^\mu \rightarrow \lambda x^\mu$  ( $x^\mu = (t, \vec{x}, z)$ ) and *inversions*  $x^\mu \rightarrow \frac{x^\mu}{x^2}$ . Of course those transformations just convey an action of  $O(2, p-1)$  on the horospheres.

The coordinates used by Maldacena in [4] are  $u \equiv \frac{R^2}{z}$ ,  $R^2$  being the square radius of the total space. In order to allow for this, we have to multiply the previous line element by  $R^2$ , getting in that way Maldacena's metric:

$$ds^2 = -\frac{R^2}{u^2} du^2 + \frac{u^2}{R^2} dx_{\parallel}^2, \quad (\text{A.50})$$

(where  $dx_{\parallel}^2$  stands for the ordinary Minkowski metric in  $M_{p-1}$ )

Horospheric coordinates break down at  $z = \infty$  ( $u = 0$ ), (which we shall call the *horizon*); which in terms of the embedding is just  $X^p = X^{p-1}$ . In terms of the global static coordinates of (A.35), this equation has solution for a given  $\tau$  for all  $\chi < \chi(\tau) \equiv \sinh^{-1} |\tan \tau|$ .

This region can be easily parametrized, using  $(X^0)^2 - \delta_{ij} X^i X^j = 1$  ( $i = 1 \dots p-2$ ) by  $X^0 = \cosh z$ ;  $X^i = n^i \sinh z$ , and the induced metric on the horizon is:

$$ds^2 = -dz^2 - \sinh^2 z d\Omega_{p-3}^2. \quad (\text{A.51})$$

In order to study the (conformal) boundary using Penrose's construction, [20] we perform the further change (from the globally static form):

$$\xi \equiv 2 \left( \tan^{-1}(e^x) - \frac{\pi}{4} \right), \quad (\text{A.52})$$

and we get

$$ds_{AdS}^2 = \cosh^2 \chi \, ds_{ESU}^2, \quad (\text{A.53})$$

where the metric of the Einstein Static Universe is given by

$$ds_{ESU}^2 = d\tau^2 - d\xi^2 - \sin^2 \xi \, d\Omega_{p-2}^2, \quad (\text{A.54})$$

(where  $-\infty \leq \tau \leq \infty$  and  $0 \leq \xi \leq \pi$ )

$CAdS$  corresponds to the portion of ESU represented by  $0 \leq \xi \leq \pi/2$ .

The boundary of  $AdS$  can be identified with the surface  $z = 0$ , which in terms of the embedding coordinates is equivalent to  $X^p - X^{p-1} = \infty$ . This sits in the ESU on  $\xi = \pi/2$ .

It is then plain that in this way of looking at things the boundary itself is part of the horizon. Let us stress once more that this latter concept is not invariant under conformal rescalings.

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